Math 246C Lecture 17 Notes

Daniel Raban

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1 Introduction to Several Complex Variables

1.1 Holomorphic functions of several complex variables

Definition 1.1. Let $\Omega \subseteq \mathbb{C}^n$ be open, and let $f : \Omega \to \mathbb{C}$ be a function $f = f(z_1, \dots, z_n) = f(x_1, y_1, \dots, x_n, y_n)$, where $z_j = x_j + y_j$. We say that f is **holomorphic** in Ω if $f \in C^1(\Omega)$ and if for every j, $z_j \mapsto f(z_1, \dots, z_j, \dots, z_n)$ where it is defined.

Define

$$\frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right)$$

for $1 \leq j \leq n$. Then f is holomorphic if and only if $f \in C^1(\Omega)$ and $\frac{\partial f}{\partial \overline{z}_j} = 0$ for all j.

Define also

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + \frac{1}{i}\frac{\partial f}{\partial y_j}\right).$$

For all $f \in C^1(\Omega)$,

$$df = \underbrace{\sum_{j=1}^n \frac{\partial f}{\partial z_j}\, dz_j}_{=:\partial f} + \underbrace{\sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j}\, d\overline{z}_j}_{=:\overline{\partial} f}.$$

So f is holomorphic iff $ba\partial f = 0$.

Example 1.1. Let $f \in L^1(\mathbb{R}^n)$ be such that $f = 0$ for large $|x|$. Then the Fourier transform

$$\widehat{f}(\xi) = \int f(x) e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^n$$

extends to the entire function

$$\widehat{f}(\zeta) = \int f(x) e^{-ix\cdot\zeta}\, dx, \qquad \zeta \in \mathbb{C}^n,$$

where $x \cdot \zeta = \sum_j x_j \zeta_j$ (in particular, there are no complex conjugates involved).

Remark 1.1. The space of holomorphic functions, $\mathrm{Hol}(\Omega)$ is a ring.

1.2 Cauchy's integral formula in a polydisc

What is the analogue of a disc in $\mathbb{C}^n$? We could try Euclidean balls, but this turns out to be more complicated.

Definition 1.2. A **polydisc** $D \subseteq \mathbb{C}^n$ si a set of the form $D = D_1 \times \cdots \times D_n$, where each D_j is an open disc in $\mathbb{C}$. The **boundary** is $\partial D = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \exists j \text{ s.t. } z_j \in \partial D_j\}$. The **distinguished boundary** of D is $\partial_0 D = \{z \in \mathbb{C}^n : z_j \in \partial D_j \ \forall j\}$.

Theorem 1.1 (Cauchy's integral formula in a polydisc)**.** *Let $D = D_1 \times \cdots D_n$ be a polydisc, let $f \in C(\overline{D})$ be such that f is separately holomorphic*[1] *in $z_j \in D_j$ for all j. Then*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{f(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n.$$

(The integral can be defined by parametrizing $\partial_0 D$: for $D_j = \{|z_j - \alpha_j| < r_j\}$, let $\zeta_j(t) = \alpha_j + r_j e^{it_j}$, $0 \leq t_j \leq 2\pi$.)

Proof. Proceed by induction on n. When $n = 1$, this is the usual Cauhy's integral formula. Suppose the formula holds for $n - 1$. Write $D = D(\alpha_1, r_1) \times D'$, where $D(\alpha_1, r_1) \subseteq \mathbb{C}$ and $D' \subseteq \mathbb{C}^{n-1}$. For every $z \in D(\alpha_1, r_1)$,

$$f(z, z') = \frac{1}{(2\pi i)^{n-1}} \int_{\partial_0 D'} \frac{f(z, \zeta')}{(\zeta_2 - z_2) \cdots (\zeta_n - z_n)} \, d\zeta'.$$

By Cauchy's integral formula and the fact that $f \in C(\overline{D})$,

$$\begin{aligned} f(z, \zeta') &= \frac{1}{2\pi i} \int_{\partial D(\alpha_1, r_1)} \frac{f(\zeta, \zeta')}{\zeta - z} \, d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial D(\alpha_1, r_1)} \frac{1}{\zeta - z} \left[\frac{1}{(2\pi i)^{n-1}} \int_{\partial_0 D'} \frac{f(z, \zeta')}{(\zeta_2 - z_2) \cdots (\zeta_n - z_n)} \, d\zeta' \right] d\zeta \\ &= \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{f(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n. \end{aligned}$$

The result follows. □

Corollary 1.1. *Let f satisfy the assumptions in the theorem. Then $f \in C^\infty(D)$, and therefore, $f \in \mathrm{Hol}(D)$.*

Corollary 1.2. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $f \in C(\Omega)$ be separately holomorphic. Then $f \in \mathrm{Hol}(\Omega)$.*

Proof. Take a polydisc D with $\overline{D} \subseteq \Omega$ around each point. □

[1] In particular, we are not assuming that f is holomorphic because we do not assume that $f \in C^1$.

1.3 Local uniform convergence of holomorphic functions

Theorem 1.2. *Let $u_k \in \mathrm{Hol}(\Omega)$ be such that $u_k \to u$ locally uniformly in Ω. Then $u \in \mathrm{Hol}(\Omega)$, and for every α, $\partial^\alpha u_k \to \partial^\alpha u$ locally uniformly. Here, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multiindex, and $\partial^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n}$.*

Proof. Let D be a polydisc with $\overline{D} \subseteq \Omega$. Then

$$u_k(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{u_k(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}\, d\zeta, \qquad z \in D.$$

It follows that $u \in \mathrm{Hol}(\Omega)$, and $\partial^\alpha u_k \to \partial^\alpha u$ uniformly in a neighborhood of the center of D for all α. □

1.4 Cauchy's estimates

Let $D \subseteq \mathbb{C}^n$ be a polydisc, let $u \in C(\overline{D}) \cap \mathrm{Hol}(D)$, and write

$$u(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{u(\zeta)}{(\zeta - z)^E}\, d\zeta.$$

Here, when α is a multiindex, write $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, and denote $E(1, \ldots, 1)$. Also, when α is a multiindex, denote $\alpha! := \alpha_1! \cdots \alpha_n!$. Then for all α,

$$\partial^\alpha u(z) = \frac{\alpha!}{(2\pi i)^n} \int_{\partial_0 D} \frac{u(\zeta)}{(\zeta - z)^{E+\alpha}}\, d\zeta.$$

We then have Cauchy's estimates:

Theorem 1.3 (Cauchy's estimates)**.** *Let $D \subseteq \mathbb{C}^n$ be a polydisc centered at 0, and let $u \in C(\overline{D}) \cap \mathrm{Hol}(D)$. Then*

$$|\partial^\alpha u(0)| \leq \alpha! \frac{M}{r^\alpha}, \qquad M = \sup_{\partial_0 D} |u|.$$

Proof. By taking derivatives in the Cauchy integral formula as above, we get

$$|\partial^\alpha u(0)| \leq \frac{\alpha!}{(2\pi i)^n} \frac{M (2\pi i)^n r^E}{r^{E\alpha}} = \alpha! \frac{M}{r^\alpha}.$$ □

1.5 Analyticity of holomorphic functions

Theorem 1.4. *Let $D \subseteq \mathbb{C}^n$ be a polydisc centered at 0, and let $f \in \mathrm{Hol}(D)$. We have, with normal convergence in D:*

$$f(z) = \sum_\alpha \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha.$$

Here, normal convergence means that $\sum u_j$ converges normally in Ω ($\sum \sup_K |u_j| < \infty$) for all compact $K \subseteq \Omega$.